DIAGONALS OF NORMAL OPERATORS WITH FINITE SPECTRUM

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Abstract. Let $X = \{\lambda_1, \dots, \lambda_N\}$ be a finite set of complex numbers and let A be a normal operator with spectrum X that acts on a separable Hilbert space H. Relative to a fixed orthonormal basis $e_1, e_2, \dots$ for H, A gives rise to a matrix whose diagonal is a sequence $d = (d_1, d_2, \dots)$ with the property that each of its terms d_n belongs to the convex hull of X. Not all sequences with that property can arise as the diagonal of a normal operator with spectrum X.

The case where X is a set of real numbers has received a great deal of attention over the years, and is reasonably well (though incompletely) understood. In this paper we take up the case in which X is the set of vertices of a convex polygon in $\mathbb{C}$. The critical sequences d turn out to be those that accumulate rapidly in X in the sense that

$$\sum_{n=1}^{\infty} \operatorname{dist}(d_n, X) < \infty.$$

We show that there is an abelian group Γ_X – a quotient of $\mathbb{R}^2$ by a countable subgroup with concrete arithmetic properties – and a surjective mapping of such sequences $d \mapsto s(d) \in \Gamma_X$ with the following property: If $s(d) \neq 0$, then d is not the diagonal of any such operator A.

We also show that while this is the only obstruction when $N = 2$, there are other (as yet unknown) obstructions when $N = 3$.

1. Introduction.

Given a self-adjoint $n \times n$ matrix A, the diagonal of A and the eigenvalue list of A are two points of $\mathbb{R}^n$ that bear some relation to each other. The Schur-Horn theorem characterizes that relation in terms of a system of linear inequalities [Sch23], [Hor54]. That characterization has attracted a great deal of interest over the years, and has been generalized in remarkable ways. For example, [Kos73], [Ati82], [GS82], [GS84] represent some of the milestones. More recently, a characterization of the diagonals of projections acting on infinite dimensional Hilbert spaces has been discovered [Kad02a],[Kad02b], and a version of the Schur-Horn theorem for positive trace-class operators was given in [AK06]. The latter reference contains a somewhat more complete historical discussion.

Let X be a finite subset of the complex plane $\mathbb{C}$, and consider the set $\mathcal{N}(X)$ of all normal operators acting on a separable Hilbert space H that

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have spectrum X with uniformly infinite multiplicity,

$$\mathcal{N}(X) = \{A \in \mathcal{B}(H) : A^*A = AA^*, \quad \sigma(A) = \sigma_e(A) = X\}.$$

The set $\mathcal{N}(X)$ is invariant under the action of the group of $*$-automorphisms of $\mathcal{B}(H)$, and it is closed in the operator norm. Fixing an orthonormal basis $e_1, e_2, \dots$ for H, one may consider the (non-closed) set $\mathcal{D}(X)$ of all diagonals of operators in $\mathcal{N}(X)$

$$\mathcal{D}(X) = \{(\langle Ae_1, e_1\rangle, \langle Ae_2, e_2\rangle, \dots) \in \ell^\infty : A \in \mathcal{N}(X)\}.$$

In this paper we address the problem of determining the elements of $\mathcal{D}(X)$.

Notice that for every sequence $d = (d_1, d_2, \dots)$ in $\mathcal{D}(X)$, *each term* d_n *must belong to the convex hull of* X. Indeed, since there is a normal operator A with spectrum X such that $d_n = \langle Ae_n, e_n\rangle$, $n \geq 1$, each d_n must belong to the numerical range of A, and the closure of the numerical range of a normal operator is the convex hull of its spectrum.

This necessary condition $d_n \in \operatorname{conv} X$, $n \geq 1$, is not sufficient. Indeed, a characterization of $\mathcal{D}(\{0,1\})$ (the set of diagonals of projections) was given in [Kad02b], the main assertion of which can be paraphrased as follows:

Theorem 1.1 (Theorem 15 of [Kad02b])**.** *Let* $d = (d_1, d_2, \dots) \in \ell^\infty$ *be a sequence satisfying* $0 \leq d_n \leq 1$ *for every* n *and*

$$\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty} 1 - d_n = \infty.$$

Let $a, b \in [0, \infty]$ *be the numbers*

$$a = \sum_{d_n \leq 1/2} d_n, \qquad b = \sum_{d_n > 1/2} 1 - d_n.$$

Then one has the following dichotomy:

(i) *If* $a + b = \infty$*, then* $d \in \mathcal{D}(\{0,1\})$*.*
(ii) *If* $a + b < \infty$*, then* $d \in \mathcal{D}(\{0,1\}) \iff a - b \in \mathbb{Z}$*.*

In a recent paper[AK06], a related spectral characterization was found for the possible diagonals of positive trace-class operators. That paper did not address the case of more general self-adjoint operators, and in particular, the results of [AK06] shed no light on the phenomenon (ii) of Theorem 1.1. This paper grew out of an effort to understand that phenomenon as an index obstruction. We achieve that for certain finite subsets $X \subseteq \mathbb{C}$ in place of $\{0,1\}$, namely those that are affinely independent in the sense that none of the points of X can be written as a nontrivial convex combination of the others - i.e., when X is the set of vertices of a convex polygon.

The basic issues taken up here bear some relation to A. Neumann's work on the infinite-dimensional Schur-Horn theorem for self-adjoint operators. But there is a fundamental difference in the nature of the characterizations of [Neu99] and the results below that goes beyond the fact that Neumann confines attention to self-adjoint operators. The comparison is clearly seen

for the two-point set $X = \{0, 1\}$. In that case, the results of [Neu99] provide the following description of the *closure* of $\mathcal{D}(X)$ in the ℓ^∞-norm:

$$\overline{\mathcal{D}(X)} = \{d = (d_n) \in \ell^\infty : 0 \leq d_n \leq 1, \quad n = 1, 2, \dots \},$$

see Lemma 2.13 and Proposition 3.12 of [Neu99]. Thus, the exceptional cases described in part (ii) of Theorem 1.1 disappear when one passes from $\mathcal{D}(X)$ to its closure in the ℓ^∞-norm. In more explicit terms, while sequences $d = (d_n)$ satisfying $0 \leq d_n \leq 1$, $n \geq 1$, $a + b < \infty$, $a - b \notin \mathbb{Z}$, fail to belong to $\mathcal{D}(X)$, they are all absorbed into its norm-closure.

It is these "exceptional" cases that we seek to understand here, for more general finite sets $X \subseteq \mathbb{C}$. Our main result (Theorem 6.1 below) identifies an index obstruction corresponding to (ii) above when X is the set of vertices of a convex polygon P. Specifically, consider the set of all sequences $d = (d_n)$ that satisfy $d_n \in P$, $n = 1, 2, \dots$, and which accumulate rapidly in X in the precise sense that

$$\sum_{n=1}^{\infty} \operatorname{dist}(d_n, X) < \infty.$$

We show that there is a discrete abelian group Γ_X, depending only on the arithmetic properties of X, and a surjective mapping $d \mapsto s(d) \in \Gamma_X$ of the set of all such sequences d, with the following property: If $s(d) \neq 0$, then d is not the diagonal of any operator in $\mathcal{N}(X)$. We use Theorem 1.1 to show that this is the only obstruction in the case of two point sets; but we also show by example that there are other (as yet unkown) obstructions in the case of three-point sets.

Finally, I want to thank Richard Kadison, whose work [Kad02a], [Kad02b] initially inspired this effort, and with whom I have had the pleasure of many helpful conversations.

2. Sequences in $\mathrm{Lim}^1(\mathrm{X})$ and the group Γ_X

Let $X = \{\lambda_1, \dots, \lambda_N\}$ be a finite set of complex numbers. For every complex number z we write

$$d(z, X) = \min_{\lambda \in X} |z - \lambda|$$

for the distance from z to X. We consider the space $\mathrm{Lim}^1(\mathrm{X})$ of all sequences $a = (a_1, a_2, \dots) \in \ell^\infty$ with the property

$$\sum_{n=1}^{\infty} d(a_n, X) < \infty. \tag{2.1}$$

Thus, a sequence $a = (a_n)$ belongs to $\mathrm{Lim}^1(\mathrm{X})$ iff all of its limit points belong to X and it converges rapidly to its limit points in the following sense: there is a sequence $x = (x_n)$ satisfying $x_n \in X$ for every $n = 1, 2, \dots$, and

$$\sum_{n=1}^{\infty} |a_n - x_n| < \infty. \tag{2.2}$$

In the context of Theorem 1.1, (2.2) reduces to the hypothesis of (ii) when $X = \{0, 1\}$, see Section 7. In this section we show that every element $a \in \mathrm{Lim}^1(\mathrm{X})$ has a "renormalized" sum that takes values in an abelian group Γ_X naturally associated with X.

For fixed $a \in \mathrm{Lim}^1(\mathrm{X})$ there are many X-valued sequences $x = (x_n)$ that satisfy (2.2). Nevertheless, one can attempt to define a "renormalized" sum of an element $a \in \mathrm{Lim}^1(\mathrm{X})$ by choosing a sequence $x_n \in X$ that satisfies (2.2) and forming the complex number

$$s = \sum_{n=1}^{\infty} a_n - x_n.$$

While the value of s depends on the choice of $x \in X$, the following observation shows that the ambiguity is associated with a countable subgroup of the additive group of $\mathbb{C}$.

Proposition 2.1. *Let $X = \{\lambda_1, \dots, \lambda_N\}$ be a finite subset of $\mathbb{C}$ and fix $a = (a_n) \in \mathrm{Lim}^1(\mathrm{X})$. For any two sequences $x = (x_n), y = (y_n)$ of points in X that satisfy*

$$\sum_{n=1}^{\infty} |a_n - x_n| < \infty, \qquad \sum_{n=1}^{\infty} |a_n - y_n| < \infty,$$

the sequence of differences $x - y = (x_n - y_n)$ is finitely nonzero, and there are integers $\nu_1, \dots, \nu_N \in \mathbb{Z}$ such that $\nu_1 + \nu_2 + \cdots + \nu_N = 0$ and

$$\sum_{n=1}^{\infty} x_n - y_n = \nu_1\lambda_1 + \nu_2\lambda_2 + \cdots + \nu_N\lambda_N. \tag{2.3}$$

Proof. Since $a-y$ and $a-x$ both belong to ℓ^1, their difference $x-y$ must also belong to ℓ^1. Since $x_n - y_n$ takes values in the finite set of differences $X - X$ and belongs to ℓ^1, it must vanish for all but finitely many n, and for each of the remaining n, $x_n - y_n$ is of the form $\lambda_{i_n} - \lambda_{j_n}$ where $i_n, j_n \in \{1, 2, \dots, N\}$. It follows that

$$\sum_{n=1}^{\infty} x_n - y_n$$

is a finite sum of terms of the form $\lambda_i - \lambda_j$, $1 \le i, j \le N$, and such a number has the form (2.3) with integer coefficients $\nu_k \in \mathbb{Z}$ having sum 0. $\square$

Definition 2.2 (The obstruction group Γ_X)**.** For every finite set of $N \ge 2$ complex numbers $X = \{\lambda_1, \dots, \lambda_N\}$, let K_X be the additive subgroup of $\mathbb{C}$ consisting of all z of the form

$$z = \nu_1\lambda_1 + \cdots + \nu_N\lambda_N$$

where $\nu_1, \dots, \nu_N \in \mathbb{Z}$ satisfy $\nu_1 + \cdots + \nu_N = 0$. Γ_X will denote the quotient of abelian groups

$$\Gamma_X = \mathbb{C}/K_X.$$

K_X is the subgroup of $\mathbb{C}$ generated by the set of differences $\lambda_i - \lambda_j$, for $i, j = 1, \dots, N$, or equivalently by $\{\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \dots, \lambda_N - \lambda_1\}$. Hence the rank of K_X is at most $N-1$. The rank is $N-1$ iff when one views $\mathbb{C}$ as a vector space over the field $\mathbb{Q}$ of rational real numbers, the set of differences $\{\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \dots, \lambda_N - \lambda_1\}$ becomes a linearly independent set.

By Proposition 2.1, we can define a map $s : \mathrm{Lim}^1(\mathrm{X}) \to \Gamma_X$ as follows: For every $a = (a_n) \in \mathrm{Lim}^1(\mathrm{X})$, choose a sequence $x = (x_n)$ that takes values in X and satisfies $\sum_n |a_n - x_n| < \infty$, and let $s(a)$ be the coset

$$s(a) = \sum_{n=1}^{\infty} a_n - x_n + K_X \in \Gamma_X.$$

Definition 2.3 (Renormalized Sum)**.** For every sequence $a \in \mathrm{Lim}^1(\mathrm{X})$, the element $s(a) \in \Gamma_X$ is called the *renormalized sum* of a.

When it is necessary to call attention to the set X of vertices, we will write $s_X(a)$ rather than $s(a)$.

Remark 2.4 (Surjectivity of the map $s : \mathrm{Lim}^1(\mathrm{X}) \to \Gamma_X$). One thinks of Γ_X as an uncountable discrete abelian group. It is easy to see that the map s is surjective. Indeed, for any $z \in \mathbb{C}$ the coset $z + K_X \in \Gamma_X$ is realized as the value $s(a)$ of a renormalized sum as follows. Choose any sequence $u = (u_n)$ in ℓ^1 such that

$$\sum_{n=1}^{\infty} u_n = z,$$

and let $x = (x_n)$ be an arbitrary sequence satisfying $x_n \in X$ for every $n \geq 1$. Then the sum $a = u + x$ belongs to $\mathrm{Lim}^1(\mathrm{X})$ and satisfies $s(a) = z + K_X$.

We require the following elementary description of sequences in $\mathrm{Lim}^1(\mathrm{X})$.

Proposition 2.5. *Let $X = \{\lambda_1, \dots, \lambda_N\}$ be a finite set of complex numbers and let $a = (a_n) \in \ell^\infty$. Then the following are equivalent:*

(i) $a \in \mathrm{Lim}^1(\mathrm{X})$.

(ii) *One has the summability condition*

$$\sum_{n=1}^{\infty} |f(a_n)| < \infty,$$

where f is the polynomial $f(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_N)$.

Proof. Let

$$\delta = \min_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|$$

be the minimum distance between distinct points of X. Note first that whenever $z \in \mathbb{C}$ satisfies $d(z, X) \leq \delta/2$, one has

$$|f(z)| \geq d(z, X)(\delta/2)^{N-1}. \tag{2.4}$$

Indeed, if we choose k such that $d(z, X) = |z - \lambda_k|$, then for $j \neq k$ we have

$$|z - \lambda_j| \geq |\lambda_k - \lambda_j| - |z - \lambda_k| \geq \delta - \delta/2 = \delta/2,$$

hence

$$|(z-\lambda_1)\cdots(z-\lambda_N)| \geq |z-\lambda_k|(\delta/2)^{N-1},$$

and (2.4) follows.

If $a = (a_1, a_2, \dots)$ is a sequence such that $\sum_n |f(a_n)|$ converges, then since $|f(z)| \geq d(z, X)^N$ for all $z \in \mathbb{C}$, it follows that $d(a_n, X)^N \to 0$ as $n \to \infty$, hence there is an n_0 such that $d(a_n, X) \leq \delta/2$ for $n \geq n_0$. (2.4) implies

$$\sum_{n=n_0}^{\infty} |f(a_n)| \geq (\delta/2)^{N-1} \sum_{n=n_0}^{\infty} d(a_n, X),$$

so that $\sum_n d(a_n, X)$ converges, hence $a \in \mathrm{Lim}^1(\mathrm{X})$.

Conversely, assuming that $\sum_n d(a_n, X)$ converges, let

$$R = \max(|\lambda_1|, \dots, |\lambda_N|) > 0.$$

Since $d(a_n, X) \to 0$, we can find n_1 such that $|a_n| \leq 2R$ for $n \geq n_1$. Choosing $k_1, k_2, \dots$ so that $d(a_n, X) = |a_n - \lambda_{k_n}|$ we have

$$|f(a_n)| = |a_n - \lambda_1| \cdots |a_n - \lambda_N| \leq |a_n - \lambda_{k_n}|(3R)^{N-1} = d(a_n, X)(3R)^{N-1}$$

for all $n \geq n_1$, hence $\sum_n |f(a_n)|$ converges. □

3. X-decompositions

We view ℓ^∞ as a commutative C^*-algebra with unit $\mathbf{1}$, and the elements of ℓ^∞ as bounded functions $a : \mathbb{N} \to \mathbb{C}$, with norm

$$\|a\| = \sup_{n \geq 1} |a(n)|.$$

Let P be a convex polygon in the complex plane and let $X = \{\lambda_1, \dots, \lambda_N\}$ be its set of vertices, with $N \geq 2$. In this section we describe an elementary (nonunique) decomposition for sequences that take values in P, and we show that under certain circumstances, all such decompositions of a sequence must share a key property.

Proposition 3.1. *Let $P \subseteq \mathbb{C}$ be a convex polygon with vertices $\{\lambda_1, \dots, \lambda_N\}$. Every sequence $a \in \ell^\infty$ satisfying $a(n) \in P$, $n \geq 1$, can be decomposed into a sum of the form*

$$a = \lambda_1 e_1 + \lambda_1 e_2 + \cdots + \lambda_N e_N \tag{3.1}$$

where $e_1, \dots, e_N$ are positive elements of ℓ^∞ satisfying $e_1 + \cdots + e_N = \mathbf{1}$.

Conversely, any sequence a of the form (3.1), with positive elements e_k summing to $\mathbf{1}$, must satisfy $a(n) \in P$ for every $n \geq 1$.

Proof. Fix a and choose $n \geq 1$. Since $a(n) \in P$ and P is the convex hull of $\{\lambda_1, \dots, \lambda_N\}$, we can find a point $(e_1(n), \dots, e_N(n)) \in \mathbb{R}^N$ such that $e_1(n) \geq 0, \dots, e_N(n) \geq 0$, $e_1(n) + \cdots + e_N(n) = 1$, and

$$a(n) = \sum_{k=1}^{N} e_k(n)\lambda_k.$$

The sequences $e_k = (e_k(1), e_k(2), \dots) \in \ell^\infty$ satisfy $e_k \geq 0$, $e_1 + \cdots + e_N = \mathbf{1}$, and the asserted representation (3.1) follows.

The converse assertion is obvious. $\square$

Definition 3.2. Let $P \subseteq \mathbb{C}$ be a convex polygon whose set of vertices is $X = \{\lambda_1, \dots, \lambda_N\}$, and let $a \in \ell^\infty$ satisfy $a(n) \in P$, $n \geq 1$. A representation of the form (3.1) is called an X-*decomposition* of a.

Despite the fact that X-decompositions are not unique except in very special circumstances, there is a property common to all X-decompositions of a in cases where

$$\sum_{n=1}^{\infty} d(a(n), X) < \infty.$$

That result (Theorem 3.4 below) requires the case $n = 2$ of the following:

Lemma 3.3. *Let P be a convex polyhedron in $\mathbb{R}^n$ with extreme points $x_1, \dots, x_r$. Consider the simplex*

$$\Delta = \{(t_1, \dots, t_r) \in \mathbb{R}^r : t_i \geq 0,\ t_1 + \cdots + t_r = 1\}$$

and the affine map of Δ onto P defined by

$$t \in \Delta \mapsto x(t) = t_1 x_1 + \cdots + t_r x_r.$$

For any choice of norms on $\mathbb{R}^r$ and $\mathbb{R}^n$, there is a constant $C > 0$ such that

$$d(t, \{\delta_1, \dots, \delta_r\}) \leq C \cdot d(x(t), \{x_1, \dots, x_r\}), \qquad t \in \Delta, \tag{3.2}$$

where $d(v, S) = \inf\{\|v - s\| : s \in S\}$ denotes the distance from a vector v to a set S, and where $\delta_1, \dots, \delta_r$ are the extreme points of Δ, $(\delta_k)_j = \delta_{kj}$.

Proof. It suffices to show that for each $k = 1, \dots, r$, there is a constant C_k such that

$$\|t - \delta_k\| \leq C_k \cdot \|x(t) - x_k\|, \qquad t \in \Delta. \tag{3.3}$$

Indeed, (3.3) implies

$$\min_{1 \leq k \leq r} \|t - \delta_k\| \leq C \min_{1 \leq k \leq r} \|x(t) - x_k\|, \qquad t \in \Delta,$$

where $C = \max(C_1, \dots, C_r)$, and (3.2) follows.

By symmetry, it suffices to prove (3.3) for $k = 1$; moreover, after performing an affine translation if necessary, there is no loss of generality if we assume that $x_1 = 0$ is one of the extreme points of P. For every extreme point e of a convex polyhedron P, there is a supporting hyperplane that meets P only at $\{e\}$. Thus there is a linear functional f on $\mathbb{R}^n$ such that $f(x) > 0$ for all nonzero $x \in P$. For each $t \in \Delta$ we have

$$f(x(t)) = \sum_{k=2}^{r} t_k f(x_k) \geq \min_{2 \leq k \leq r} f(x_k) \cdot \sum_{k=2}^{r} t_k.$$

After noting that $f(x(t)) \leq \|f\| \cdot \|x(t)\|$, we obtain

$$\|t - \delta_k\| = \sum_{k=1}^{r} |t_k - \delta_1(k)| = 2\sum_{k=2}^{r} t_k \leq \frac{2 \cdot \|f\|}{\min(f(x_2), \dots, f(x_r))} \cdot \|x(t)\|,$$

and (3.3) follows after noting that the left side of the preceding inequality dominates $\epsilon \cdot \|t - \delta_1\|$ for an appropriately small positive constant ϵ. $\square$

Theorem 3.4. *Let $X = \{\lambda_1, \dots, \lambda_N\}$ be the set of vertices of a convex polygon $P \subseteq \mathbb{C}$ and let $a \in \ell^\infty$ satisfy $a(n) \in P$ for $n \geq 1$. If $a \in \mathrm{Lim}^1(\mathrm{X})$, then for every X-decomposition of the form (3.1)*

$$a = \lambda_1 e_1 + \dots + \lambda_N e_N,$$

each of the sequences $e_1, \dots, e_N$ belongs to $\mathrm{Lim}^1\{0, 1\}$.

Proof. Consider the Euclidean norm on $\mathbb{C}$, the norm

$$\|(x_1, \dots, x_N)\| = |x_1| + \dots + |x_N|$$

on $\mathbb{R}^N$, and fix $n = 1, 2, \dots$. Since the point $(e_1(n), \dots, e_N(n)) \in \mathbb{R}^N$ belongs to the simplex Δ of Lemma 3.3, there is a constant $C > 0$ such that

$$\begin{aligned} d((e_1(n), \dots, e_N(n)), \{\delta_1, \dots, \delta_N\}) &\leq C \cdot d(a(n), \{\lambda_1, \dots, \lambda_N\}) \\ &= C \cdot d(a(n), X), \qquad n \geq 1. \end{aligned}$$

Since for every point $t = (t_1, \dots, t_N)$ in the simplex Δ and for every fixed $k = 1, \dots, N$ we have

$$\begin{aligned} d(t_k, \{0, 1\}) &= \min(t_k, 1 - t_k) \\ &\leq \min_{1 \leq k \leq N} \Big(\sum_{j \neq k} t_j + (1 - t_k)\Big) = d(t, \{\delta_1, \dots, \delta_N\}), \end{aligned}$$

it follows that

$$\begin{aligned} d(e_k(n), \{0, 1\}) &\leq d((e_1(n), \dots, e_N(n)), \{\delta_1, \dots, \delta_N\}) \\ &\leq C d(a(n), \{\lambda_1, \dots, \lambda_N\}) = C d(a(n), X). \end{aligned}$$

Using $a \in \mathrm{Lim}^1(\mathrm{X})$, we can sum the preceding inequality on n to obtain

$$\sum_{n=1}^{\infty} d(e_k(n), \{0, 1\}) \leq C \cdot \sum_{n=1}^{\infty} d(a(n), X) < \infty.$$

Hence $e_k \in \mathrm{Lim}^1\{0, 1\}$. $\square$

4. Two Projections.

It is known that for any pair of projections $P, Q \in \mathcal{B}(H)$ for which $P - Q$ is a trace-class operator, $\mathrm{trace}(P - Q)$ must be an integer. For example, the result can be found in Effros' article ([Eff89], see Lemma 4.1). We require an appropriate extension of that result to the case where $P - Q$ is merely a Hilbert-Schmidt operator, Theorem 4.1 below. Throughout the remainder of this paper, we write $\mathcal{L}^1$ (resp. $\mathcal{L}^2$) for the Banach space of

trace-class operators (resp. Hilbert-Schmidt operators) acting on a given Hilbert space, and we write $P^\perp$ for $\mathbf{1} - P$ when $P \in \mathcal{B}(H)$ is a projection.

Theorem 4.1. *Let M, N be subspaces of a Hilbert space H with respective projections P, Q, and assume that $P - Q \in \mathcal{L}^2$.*

Then both $Q(P-Q)Q$ and $Q^\perp(P-Q)Q^\perp$ belong to $\mathcal{L}^1$, both subspaces $M \cap N^\perp$ and $N \cap M^\perp$ are finite-dimensional, and

$$\operatorname{trace}(Q(P-Q)Q + Q^\perp(P-Q)Q^\perp) = \dim(M \cap N^\perp) - \dim(N \cap M^\perp). \tag{4.1}$$

In particular, $QPQ + Q^\perp PQ^\perp - Q$ is a trace-class operator such that

$$\operatorname{trace}(QPQ + Q^\perp PQ^\perp - Q) \in \mathbb{Z}. \tag{4.2}$$

In the proof we will show that the left side of (4.1) is the index of a Fredholm operator, and for that we require the following elementary result for which we lack a convenient reference:

Lemma 4.2. *Let H, K be Hilbert spaces and let $A : H \to K$ be an operator such that both $\mathbf{1}_H - A^*A$ and $\mathbf{1}_K - AA^*$ are trace-class. Then A is a Fredholm operator in $\mathcal{B}(H, K)$ whose index is given by the formula*

$$\operatorname{index} A = \operatorname{trace}(\mathbf{1}_H - A^*A) - \operatorname{trace}(\mathbf{1}_K - AA^*) \tag{4.3}$$

Proof. Consider the polar decomposition $A = UB$, where B is a positive operator and U is a partial isometry with initial space $\ker A^\perp$ and range $\overline{\operatorname{ran} A} = \ker A^{*\perp}$. Let C be the restriction of B^2 to $\ker A^\perp$. Then $A^*A = B^2 = C \oplus 0_{\ker A}$ and $AA^* = C' \oplus 0_{\ker A^*}$, where C' is unitarily equivalent to C; indeed, the restriction of U to $\ker A^\perp$ implements a unitary equivalence of C and C'.

It follows that $\mathbf{1} - A^*A = (\mathbf{1} - C) \oplus \mathbf{1}_{\ker A}$, and $\mathbf{1} - AA^*$ is unitarily equivalent to $(\mathbf{1} - C) \oplus \mathbf{1}_{\ker A^*}$. Since $\mathbf{1} - C$ is a trace-class operator, we have $\operatorname{trace}(\mathbf{1} - A^*A) = \operatorname{trace}(\mathbf{1} - C) + \dim\ker A$, and similarly $\operatorname{trace}(\mathbf{1} - AA^*) = \operatorname{trace}(\mathbf{1} - C) + \dim\ker A^*$. The terms involving $\operatorname{trace}(\mathbf{1} - C)$ cancel, and

$$\operatorname{trace}(\mathbf{1} - A^*A) - \operatorname{trace}(\mathbf{1} - AA^*) = \dim\ker A - \dim\ker A^* = \operatorname{index} A,$$

as asserted. □

Proof of Theorem 4.1. We claim first that $Q(P-Q)Q \in \mathcal{L}^1$. Indeed, we have $Q(P-Q)Q = -(Q - QPQ)$, and $Q - QPQ$ is a positive operator satisfying

$$\operatorname{trace}(Q - QPQ) = \operatorname{trace} QP^\perp Q = \operatorname{trace} |P^\perp Q|^2 = \operatorname{trace} |(P-Q)Q|^2 < \infty.$$

Similarly, $Q^\perp(P-Q)Q^\perp = Q^\perp PQ^\perp$, and

$$\operatorname{trace} Q^\perp PQ^\perp = \operatorname{trace} |PQ^\perp|^2 = \operatorname{trace} |(P-Q)Q^\perp|^2 < \infty,$$

so that $Q^\perp(P-Q)Q^\perp \in \mathcal{L}^1$.

Let H_0 be the subspace of H spanned by the mutually orthogonal subspaces $M \cap N^\perp$ and $N \cap M^\perp$. The restriction of $P - Q$ to H_0 is unitary with eigenvalues ± 1, hence $\dim H_0 = \operatorname{trace} |(P-Q)P_{H_0}|^2 \le \operatorname{trace} |P-Q|^2$ must be finite.

Consider the operator $A : N \to M$ defined by restricting P to $N = QH$. Obviously, $\ker A = N \cap M^\perp$ and $\ker A^* = M \cap N^\perp$, and we claim that A satisfies the hypotheses of Lemma 4.2. Indeed,

$$(\mathbf{1}_N - A^*A)Q = Q - QPQ = QP^\perp Q = |P^\perp Q|^2$$
$$(\mathbf{1}_M - AA^*)P = P - PQP = PQ^\perp P = |Q^\perp P|^2.$$

Since $P - Q$ is Hilbert-Schmidt, $Q^\perp P = (P - Q)P$ and $P^\perp Q = (Q - P)Q$ are both Hilbert-Schmidt, hence $|Q^\perp P|^2$ and $|P^\perp Q|^2$ are both trace-class. Thus we can apply (4.3) to obtain

$$\begin{aligned}\operatorname{index} A &= \operatorname{trace} PQ^\perp P - \operatorname{trace} QP^\perp Q = \operatorname{trace} Q^\perp PQ^\perp - \operatorname{trace} QP^\perp Q \\ &= \operatorname{trace} Q^\perp(P - Q)Q^\perp + \operatorname{trace} Q(P - Q)Q.\end{aligned}$$

The left side is $\dim\ker A - \dim\ker A^* = \dim(N \cap M^\perp) - \dim(M \cap N^\perp)$, and (4.1) follows from the preceding formula. $\square$

5. Projections with diagonals in $\mathrm{Lim}^1\{0, 1\}$.

In this section we characterize the projections in $\mathcal{B}(H)$ whose diagonals relative to a given orthonormal basis belong to $\mathrm{Lim}^1\{0, 1\}$.

Proposition 5.1. *Let $e_1, e_2, \dots$ be an orthonormal basis for a Hilbert space H, let $\mathcal{A}$ be the maximal abelian von Neumann algebra of all operators that are diagonalized by (e_n), and let $E : \mathcal{B}(H) \to \mathcal{A}$ be the trace-preserving conditional expectation*

$$E(A) = \sum_{n=1}^{\infty} \langle Ae_n, e_n\rangle e_n \otimes \bar{e_n}. \tag{5.1}$$

For every projection $P \in \mathcal{B}(H)$, the following are equivalent:
(i) *The diagonal of P relative to the basis (e_n) belongs to* $\mathrm{Lim}^1\{0, 1\}$.
(ii) $E(P) - E(P)^2 \in \mathcal{L}^1$.
(iii) $P \in \mathcal{A} + \mathcal{L}^2$.

The proof of Proposition 5.1 requires the following formula:

Lemma 5.2. *Let $E : \mathcal{B}(H) \to \mathcal{A}$ be the map (5.1). Then for every projection $P \in \mathcal{B}(H)$ we have*

$$\operatorname{trace}(P - E(P))^2 = \operatorname{trace}(E(P) - E(P)^2). \tag{5.2}$$

Proof of Lemma 5.2. We have $E(P)e_n = d_n e_n$, where $d_n = \langle Pe_n, e_n\rangle$. Since $d_n e_n$ is the projection of Pe_n onto the one-dimensional space $\mathbb{C} \cdot e_n$, we have

$$\|Pe_n - d_n e_n\|^2 = \|Pe_n\|^2 - \|d_n e_n\|^2 = \langle Pe_n, e_n\rangle - d_n^2 = d_n - d_n^2.$$

Hence

$$\operatorname{trace}[P - E(P)]^2 = \sum_{n=1}^{\infty} \|Pe_n - d_n e_n\|^2 = \sum_{n=1}^{\infty} d_n - d_n^2,$$

and the right side is evidently the trace of $E(P) - E(P)^2$. $\square$

Proof of Proposition 5.1. Let $d = (d_1, d_2, \dots)$ be the diagonal of P relative to (e_n), $d_n = \langle Pe_n, e_n\rangle$, $n \geq 1$. Then

$$\operatorname{trace}(E(P) - E(P)^2) = \sum_{n=1}^{\infty} d_n - d_n^2,$$

hence the equivalence of (i) and (ii) follows from Proposition 2.5.

(iii) $\implies$ (ii): Assume first that the projection P can be decomposed into a sum $P = A + T$ where $A \in \mathcal{A}$ and T is Hilbert-Schmidt. Then $P - E(P) = T - E(T)$, and $T - E(T)$ is a Hilbert-Schmidt operator. By (5.2), we obtain

$$\operatorname{trace}(E(P) - E(P)^2) = \operatorname{trace}(P - E(P))^2 = \operatorname{trace}(T - E(T))^2 < \infty.$$

(ii) $\implies$ (iii): Assume that $\operatorname{trace}(E(P) - E(P)^2)) < \infty$, and consider the operator $T = P - E(P)$. By (5.2), we have

$$\operatorname{trace} T^2 = \operatorname{trace}(E(P) - E(P)^2) < \infty,$$

so that T is Hilbert-Schmidt. Thus $P = E(P) + T \in \mathcal{A} + \mathcal{L}^2$. □

6. Diagonals of operators in $\mathcal{N}(X)$.

We are now in position to prove our main result. Let $X = \{\lambda_1, \dots, \lambda_N\}$ be the set of vertices of a convex polygon $P \subseteq \mathbb{C}$ and let $\mathcal{N}(X)$ be the set of all normal operators A acting on a separable Hilbert space H that have spectrum X with infinite multiplicity

$$\sigma(A) = \sigma_e(A) = X.$$

Fix an orthonormal basis $e_1, e_2, \dots$ for H. There are two necessary conditions that a sequence $d = (d_1, d_2, \dots) \in \ell^\infty$ must satisfy for it to be the diagonal of an operator $A \in \mathcal{N}(X)$, $d_n = \langle Ae_n, e_n\rangle$, $n \geq 1$, namely:

(i) $d_n \in P$ for every $n \geq 1$,
(ii) d has an X-decomposition

$$d = \lambda_1 E_1 + \cdots + \lambda_N E_N$$

in which $\sum_{n=1}^{\infty} E_k(n) = \infty$ for every $k = 1, \dots, N$.

Indeed the projections P_k arising from the spectral representation of A

$$A = \lambda_1 P_1 + \cdots + \lambda_N P_N$$

have diagonals $E_k(n) = \langle P_k e_n, e_n\rangle$ that give rise to an X-decomposition with the property (ii). The requirements (i), (ii) on a sequence do not guarantee that it is the diagonal of an operator in $\mathcal{N}(X)$. We now identify an obstruction that emerges when $d \in \operatorname{Lim}^1(\mathrm{X})$ and which involves the renormalized sum $s : \operatorname{Lim}^1(\mathrm{X}) \to \Gamma_X$ of Definition 2.2. .

Theorem 6.1. *Let $X = \{\lambda_1, \dots, \lambda_N\}$ be the set of vertices of a convex polygon $P \subseteq \mathbb{C}$ and let $d = (d_1, d_2, \dots)$ be a sequence of complex numbers satisfying $d_n \in P$, $n \geq 1$, together with the summability condition*

$$\sum_{n=1}^{\infty} |f(d_n)| < \infty, \tag{6.1}$$

where $f(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_N)$. Then $d \in \mathrm{Lim}^1(\mathrm{X})$; and if d is the diagonal of an operator in $\mathcal{N}(X)$, then $s(d) = 0$.

Proof. By Proposition 2.5, the summability condition (6.1) characterizes sequences in $\mathrm{Lim}^1(\mathrm{X})$.

Fix an orthonormal basis $e_1, e_2, \dots$ for a Hilbert space H, and assume that there is an operator $A \in \mathcal{N}(X)$ such that $d_n = \langle Ae_n, e_n \rangle$, $n = 1, 2, \dots$. In order to show that $s(d) = 0$, we must find a sequence (b_n) that takes values in X, satisfies $\sum_n |d_n - b_n| < \infty$, and we must exhibit integers $\nu_1, \dots, \nu_n$ satisfying $\nu_1 + \cdots + \nu_N = 0$, and

$$\sum_{n=1}^{\infty} d_n - b_n = \nu_1 \lambda_1 + \cdots + \nu_N \lambda_N.$$

For that, consider the maximal abelian algebra $\mathcal{A}$ of all operators that are diagonalized by the basis $e_1, e_2, \dots$, let $E : \mathcal{B}(H) \to \mathcal{A}$ be the trace-preserving conditional expectation

$$E(T) = \sum_{n=1}^{\infty} \langle Te_n, e_n \rangle e_n \otimes \overline{e_n},$$

and let $D = E(A) \in \mathcal{A}$ be the operator

$$D = \sum_{n=1}^{\infty} d_n \, e_n \otimes \overline{e_n} = \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle e_n \otimes \overline{e_n}.$$

We must find an operator $B \in \mathcal{A} \cap \mathcal{N}(X)$ such that $D - B \in \mathcal{L}^1$, and integers $\nu_1, \dots, \nu_N$ summing to zero, such that

$$\operatorname{trace}(D - B) = \nu_1 \lambda_1 + \cdots + \nu_N \lambda_N. \tag{6.2}$$

The latter are achieved as follows. Let

$$A = \lambda_1 P_1 + \cdots + \lambda_N P_N$$

be the spectral representation of A, with $P_1, \dots, P_N$ a set of mutually orthogonal infinite rank projections with sum $\mathbf{1}$. Then we have

$$D = E(A) = \lambda_1 E(P_1) + \cdots + \lambda_N E(P_N),$$

so that $E(P_1), \dots, E(P_N)$ define an X-decomposition of D. Theorem 3.4 implies that when one views the operators $E(P_k)$ as sequences in ℓ^∞, one has $E(P_k) \in \mathrm{Lim}^1(\{0, 1\})$ for each $k = 1, \dots, N$.

We claim that there is a sequence $Q_1, \dots, Q_N$ of mutually orthogonal projections in $\mathcal{A}$ having sum $\mathbf{1}$ which satisfy

$$E(P_k) - Q_k \in \mathcal{L}^1, \qquad k = 1, \dots, N. \tag{6.3}$$

Indeed, since $E(P_k) \in \mathrm{Lim}^1(\{0,1\})$ for each k, the definition of $\mathrm{Lim}^1(\{0,1\})$ implies that we can find projections $Q_1^0, \dots, Q_N^0 \in \mathcal{A}$ such that

$$E(P_k) - Q_k^0 \in \mathcal{L}^1, \qquad k = 1, \dots, N.$$

Considering each Q_k^0 as a sequence in ℓ^∞ that takes values in $\{0,1\}$, the sum $Q_1^0 + \cdots + Q_N^0$ is a sequence taking values in $\{0, 1, 2, \dots, N\}$. Consider the set $S = \{n \in \mathbb{N} : \sum_{k=1}^N Q_k^0(n) = 1\} \subseteq \mathbb{N}$. Since $P_1 + \cdots + P_N = \mathbf{1}$ we have $E(P_1) + \cdots + E(P_N) = \mathbf{1}$, and hence

$$\mathbf{1} - \sum_{k=1}^N Q_k^0 = \sum_{k=1}^N (E(P_k) - Q_k^0) \in \mathcal{L}^1.$$

It follows that

$$\sum_{n \notin S} |1 - \sum_{k=1}^N Q_k^0(n)| < \infty.$$

The latter implies that $\mathbb{N} \setminus S$ is a finite set, and that $Q_1^0 \cdot \chi_S, \dots, Q_N^0 \cdot \chi_S$ are mutually orthogonal projections with sum χ_S. Thus if we modify the sequence $Q_1^0, \dots, Q_N^0$ as follows,

$$Q_1 = Q_1^0 \cdot \chi_S + \chi_{\mathbb{N} \setminus S},\ Q_2 = Q_2^0 \cdot \chi_S,\ \dots,\ Q_N = Q_N^0 \cdot \chi_S,$$

we obtain a new sequence of projections $Q_1, \dots, Q_N \in \mathcal{A}$ which are mutually orthogonal, have sum $\mathbf{1}$, and satisfy (6.3).

Note too that since $\operatorname{trace} P_k = \operatorname{rank} P_k = \infty$ for every k, (6.3) implies that $\operatorname{trace} Q_k = \operatorname{rank} Q_k = \infty$ as well. It follows that the operator

$$B = \lambda_1 Q_1 + \cdots + \lambda_N Q_N$$

belongs to $\mathcal{A}$, satisfies $\sigma(B) = \sigma_e(B) = X$, and by construction,

$$D - B = E(A) - B \in \mathcal{L}^1. \tag{6.4}$$

It remains to show that $\operatorname{trace}(D - B)$ satisfies (6.2) for integers ν_k as described there. Indeed, since $D - B = \sum_k \lambda_k (E(P_k) - Q_k)$ and $E(P_k) - Q_k$ belongs to $\mathcal{L}^1$, we have

$$\operatorname{trace}(D - B) = \sum_{k=1}^N \lambda_k \cdot \operatorname{trace}(E(P_k) - Q_k),$$

so it suffices to show that

$$\operatorname{trace}(E(P_k) - Q_k) \in \mathbb{Z}, \qquad k = 1, \dots, N, \tag{6.5}$$

and that the sum of the N integers of (6.5) is 0.

In order to prove (6.5) we appeal to Theorem 4.1. Note first that $P-E(P)$ belongs to $\mathcal{L}^2$. Indeed, since $E(P_k)-Q_k$ is trace-class and Q_k is a projection, we have $\operatorname{trace}(E(P)-E(P)^2)<\infty$, so by (5.2),

$$\operatorname{trace}[P-E(P)]^2 = \operatorname{trace}(E(P)-E(P)^2) < \infty.$$

Since $E(P_k)-Q_k \in \mathcal{L}^1 \subseteq \mathcal{L}^2$, we obtain

$$P_k - Q_k = (P_k - E(P_k)) + E(P_k) - Q_k \in \mathcal{L}^2, \qquad k = 1, \dots, N.$$

From Theorem 4.1 we conclude that

$$Q_k P_k Q_k + Q_k^\perp P_k Q_k^\perp - Q_k \in \mathcal{L}^1,$$

and moreover

$$\nu_k = \operatorname{trace}(Q_k P_k Q_k + Q_k^\perp P_k Q_k^\perp - Q_k) \in \mathbb{Z}.$$

Since $E(\mathcal{L}^1) \subseteq \mathcal{L}^1 \cap \mathcal{A}$ and since

$$\begin{aligned} E(Q_k P_k Q_k + Q_k^\perp P_k Q_k^\perp - Q_k) &= Q_k E(P_k) Q_k + Q_k^\perp E(P_k) Q_k^\perp - Q_k \\ &= E(P_k) - Q_k, \end{aligned}$$

we find that $E(P_k) - Q_k \in \mathcal{L}^1$ and

$$\operatorname{trace}(E(P_k) - Q_k) = \operatorname{trace}(Q_k P_k Q_k + Q_k^\perp P_k Q_k^\perp - Q_k) = \nu_k \in \mathbb{Z}.$$

Since $\sum_{k=1}^N (E(P_k) - Q_k) = E(\mathbf{1}) - \mathbf{1} = 0$, we have $\nu_1 + \cdots + \nu_N = 0$. Finally,

$$\operatorname{trace}(D-B) = \sum_{k=1}^N \lambda_k \cdot \operatorname{trace}(E(P_k) - Q_k) = \sum_{k=1}^N \lambda_k \nu_k,$$

and (6.2) follows. □

7. Concluding Remarks, and an example

We point out that Theorem 6.1 specializes to the assertion (ii) $\Rightarrow$ of Theorem 1.1 in the case $X = \{0,1\}$. Indeed, a straightforward calculation shows that for the two-point set $X = \{0,1\}$ one has $K_{\{0,1\}} = \mathbb{Z}$, so that $\Gamma_{\{0,1\}} = \mathbb{C}/\mathbb{Z} = \mathbb{T} \times \mathbb{R}$. Now the hypothesis of Theorem 1.1 (ii) is that $a + b < \infty$, where a and b are defined by

$$a = \sum_{d_n \le 1/2} d_n, \qquad b = \sum_{d_n > 1/2} 1 - d_n.$$

Let (x_n) be the sequence $x_n = 0$ when $d_n \le 1/2$ and $x_n = 1$ when $d_n > 1/2$. Then $a + b$ is finite $\iff \sum_n |d_n - x_n| < \infty \iff d \in \operatorname{Lim}^1(\{0,1\})$. Moreover, $a - b = \sum_n d_n - x_n$, so that $a - b \in \mathbb{Z} \iff s_{\{0,1\}}(a) = 0$.

For more general sets X, the converse of Theorem 6.1 would assert:

Let X be the set of vertices of a convex polygon and let d be a sequence in $\operatorname{Lim}^1(\mathrm{X})$ *such that $s_X(d) = 0$. Then there is an operator $N \in \mathcal{N}(X)$ and an orthonormal basis $e_1, e_2, \dots$ for H such that $d_n = \langle N e_n, e_n \rangle$, $n \ge 1$.*

We first point out that this converse is true when X consists of just two points. To sketch the argument briefly, suppose $X = \{\lambda_1, \lambda_2\}$ with $\lambda_1 \ne \lambda_2$.

One can find an affine bijection $z \mapsto az + b$ of $\mathbb{C}$ that carries λ_1 to 0 and λ_2 to 1, and which therefore carries sequences in $\mathrm{Lim}^1(\mathrm{X})$ to sequences in $\mathrm{Lim}^1(\{0,1\})$. After noting that the operator mapping $T \mapsto aT + b\mathbf{1}$ carries $\mathcal{N}(X)$ to $\mathcal{N}(\{0,1\})$, one can make use of the implication $\Leftarrow$ of Theorem 1.1 (ii) in a straightforward way to deduce the required result.

On the other hand, the following example shows that this converse of Theorem 6.1 fails for three-point sets.

Proposition 7.1. *Let $X = \{0, 1, i\}$, i denoting $\sqrt{-1}$, and consider the sequence*

$$d = (\frac{1}{2}, \frac{i}{2}, \frac{1+i}{2}, 0, 1, i, 0, 1, i, 0, 1, i, \dots) \in \ell^\infty.$$

Then d belongs to $\mathrm{Lim}^1(\mathrm{X})$ and the renormalized sum $s_X(d)$ vanishes. But there is no normal operator $N \in \mathcal{N}(X)$ whose diagonal relative to some orthonormal basis is d.

Before giving the proof, we recall that a doubly stochastic $n \times n$ matrix $A = (a_{ij})$ is said to be *orthostochastic* if there is a unitary $n \times n$ matrix (u_{ij}) such that $a_{ij} = |u_{ij}|^2$, $1 \le i, j \le n$. We will make use of the following known example of a doubly stochastic matrix that is not orthostochastic [Hor54].

Lemma 7.2. *The 3×3 matrix*

$$(a_{ij}) = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

is not orthostochastic.

Proof. Indeed, if there were a unitary 3×3 matrix $U = (u_{ij})$ such that $|u_{ij}|^2 = a_{ij}$ for all ij, then U must have the form

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} a & 0 & b \\ 0 & c & d \\ e & f & 0 \end{pmatrix}$$

with complex entries satisfying $|a| = |b| = |c| = |d| = |e| = |f| = 1$. But the rows of such a matrix cannot be mutually orthogonal. $\square$

Proof of Proposition 7.1. Straightforward computations (that we omit) show that for the set $X = \{0, 1, i\}$, the group K_X and the obstruction group Γ_X are given by

$$K_X = \mathbb{Z} + \mathbb{Z} \cdot i, \qquad \Gamma_X = \mathbb{C}/K_X \cong \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \cong \mathbb{T}^2.$$

Let x be the sequence

$$x = (0, 0, 0, 0, 1, i, 0, 1, i, 0, 1, i, \dots).$$

Obviously $x_n \in X$ for every $n = 1, 2, \dots$, $d_n = x_n$ except for $n = 1, 2, 3$, and

$$\sum_{n=1}^{\infty} d_n - x_n = \frac{1}{2} + \frac{i}{2} + \frac{1+i}{2} = 1 + i \in \mathbb{Z} + \mathbb{Z} \cdot i = K_X.$$

Hence $d \in \mathrm{Lim}^1(\mathrm{X})$ and $s_X(d) = 0$.

Every operator $N \in \mathcal{N}(X)$ has the form $N = P + iQ$, where P, Q are mutually orthogonal infinite rank projections such that $\mathbf{1} - (P+Q)$ has infinite rank. Assuming that there is such an operator N whose matrix relative to some orthonormal basis $e_1, e_2, \dots$ has diagonal $d = (d_1, d_2, \dots)$, we argue to a contradiction as follows. Let $p, q \in \ell^\infty$ be the real and imaginary parts of the sequence d

$$p = (\frac{1}{2}, 0, \frac{1}{2}, 0, 1, 0, 0, 1, 0, 0, 1, 0, \dots)$$
$$q = (0, \frac{1}{2}, \frac{1}{2}, 0, 0, 1, 0, 0, 1, 0, 0, 1, \dots).$$

Since d is the diagonal of $P + iQ$, one may equate real and imaginary parts to obtain $p_n = \langle Pe_n, e_n\rangle$ and $q_n = \langle Qe_n, e_n\rangle$, $n = 1, 2, \dots$.

Now for $n \geq 4$, both p_n and q_n are $\{0, 1\}$-valued. Since P and Q are projections, it follows that

$$Pe_n = p_n e_n, \quad Qe_n = q_n e_n, \qquad n \geq 4,$$

and in particular, both P and Q leave the closed linear span $[e_4, e_5, e_6, \dots]$ invariant. Hence they leave its orthocomplement $[e_1, e_2, e_3]$ invariant as well. Let P_0, Q_0 be the restrictions of P, Q, respectively, to $H_0 = [e_1, e_2, e_3]$. P_0 and Q_0 are mutually orthogonal projections, and the diagonals of their matrices relative to the orthonormal basis e_1, e_2, e_3 are respectively

$$(\frac{1}{2}, 0, \frac{1}{2}), \quad \text{and } (0, \frac{1}{2}, \frac{1}{2}).$$

Each of these two diagonals has sum 1, hence P_0 and Q_0 are one-dimensional. Moreover, $R_0 = \mathbf{1}_{H_0} - (P_0 + Q_0)$ is a one-dimensional projection in $\mathcal{B}(H_0)$ whose diagonal relative to the basis e_1, e_2, e_3 is

$$(\frac{1}{2}, \frac{1}{2}, 0).$$

Hence the 3×3 matrix whose rows are the diagonals of the three projections P_0, Q_0, R_0 takes the form

$$A = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

If we now choose unit vectors f_1, f_2, f_3 so that $P_0 = [f_1]$, $Q_0 = [f_2]$ and $R_0 = [f_3]$, we find that f_1, f_2, f_3 is a second orthonormal basis for H_0, and

$$A = \begin{pmatrix} |\langle e_1, f_1\rangle|^2 & |\langle e_2, f_1\rangle|^2 & |\langle e_3, f_1\rangle|^2 \\ |\langle e_1, f_2\rangle|^2 & |\langle e_2, f_2\rangle|^2 & |\langle e_3, f_2\rangle|^2 \\ |\langle e_1, f_3\rangle|^2 & |\langle e_2, f_3\rangle|^2 & |\langle e_3, f_3\rangle|^2 \end{pmatrix} = (|u_{ij}|^2),$$

where (u_{ij}) is a unitary 3×3 matrix. This contradicts Lemma 7.2. $\square$

Proposition 7.1 shows that the necessary condition $s_X(d) = 0$ is not sufficient for a sequence $d \in \mathrm{Lim}^1(\mathrm{X})$ to be the diagonal of an operator in $\mathcal{N}(X)$ when X contains more than two points. Moreover, the precise nature of the remaining obstructions when X consists of three non-colinear points remains mysterious.

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